

# Nonextensive and superstatistical generalizations of random-matrix theory

A.Y. Abul-Magd

*Faculty of Engineering Sciences, Sinai University, El Arish, Egypt*

(Dated: February 17, 2009)

## Abstract

Random matrix theory (RMT) is based on two assumptions: (1) matrix-element independence, and (2) base invariance. Most of the proposed generalizations keep the first assumption and violate the second. Recently, several authors presented other versions of the theory that keep base invariance on the expense of allowing correlations between matrix elements. This is achieved by starting from non-extensive entropies rather than the standard Shannon entropy, or following the basic prescription of the recently suggested concept of superstatistics. We review these generalizations of RMT and illustrate their value by calculating the nearest-neighbor-spacing distributions and comparing the results of calculation with experiments and numerical-experiments on systems in transition from order to chaos

## I. INTRODUCTION

In classical mechanics, integrable Hamiltonian dynamics is characterized by the existence of as many conserved quantities as degrees of freedom. Each trajectory in the corresponding phase space evolves on an invariant hyper-torus [1, 2]. In contrast, chaotic systems are ergodic; almost all orbits fill the energy shell in a uniform way. Physical systems with integrable and fully chaotic dynamics are exceptional. A typical Hamiltonian system shows a mixed phase space in which regions of regular motion and chaotic dynamics coexist. Systems of this kind are known as mixed systems. Their dynamical behavior is by no means universal. If we perturb an integrable system, most of the periodic orbits on tori with rational frequencies disappear. However, some of these orbits persist. Elliptic periodic orbits appear surrounded by islands. They correspond to librational motions around these periodic orbits and reflect their stability. The Kolmogorov-Arnold (KAM) theorem establishes the stability with respect to small perturbations of invariant tori with a sufficiently incommensurate frequency vector. When the perturbation increases, numerical simulations show that more and more tori are destroyed. For large enough perturbations, there are locally no tori in the considered region of phase-space. The break-up of invariant tori leads to a loss of stability of the system, to chaos. Different scenarios of transition to chaos in dynamical systems have been considered. There are three main scenarios of transition to global chaos in finite-dimensional (non-extended) dynamical systems: via the cascade of period-doubling bifurcations, the Lorenz system-like transition via Hopf and Shil'nikov bifurcations, and the transition to chaos via intermittences [3, 4, 5]. It is natural to expect that there could be other (presumably many more) such scenarios in extended (infinite-dimensional) dynamical systems.

In quantum mechanics, the specification of a wave function is always related to a certain basis. In integrable systems eigenbasis of the Hamiltonian is known in principle. In this basis, each eigenfunction has just one component that obviously indicates the absence of complexity. In the nearly ordered regime, mixing of quantum states belonging to adjacent levels can be ignored and the energy levels are uncorrelated. The level-spacing distribution function obeys the Poissonian,  $\exp(-s)$ , where  $s$  is the energy spacing between adjacent levels normalized by the mean level spacing  $D$ . On the other hand, the eigenfunctions a Hamiltonian with a chaotic classical limit is unknown in principle. In other words, there is

no special basis to express the eigenstates of a chaotic system. If we try to express the wave functions of a chaotic system in terms of a given basis, their components become on average uniformly distributed over the whole basis. They are also extended in all other bases. For example, Berry [6] conjectured that the wavefunctions of chaotic quantum systems can be represented as a formal sum over elementary solutions of the Laplace equation in which real and imaginary parts of coefficients are independent identically-distributed Gaussian random variables with zero mean and variance computed from the normalization. Bohigas et al. [7] put forward a conjecture (strongly supported by accumulated numerical evidence) that the spectral statistics of chaotic systems follow random-matrix theory (RMT) [8, 9]. This theory models a chaotic system by an ensemble of random Hamiltonian matrices  $\mathbf{H}$  that belong to one of the three universal classes, orthogonal, unitary and symplectic. The theory is based on two main assumptions: the matrix elements are independent identically-distributed random variables, and their distribution is invariant under unitary transformations. These lead to a Gaussian probability density distribution for the matrix elements. The Gaussian distribution is also obtained by maximizing the Shannon entropy under constraints of normalization and existence of the expectation value of  $\text{Tr}(\mathbf{H}^\dagger \mathbf{H})$ , where  $\text{Tr}$  denotes the trace and  $\mathbf{H}^\dagger$  stands for the Hermitian conjugate of  $\mathbf{H}$  [8, 10]. The statistical information about the eigenvalues and/or eigenvectors of the matrix can be obtained by integrating out all the undesired variables from distribution of the matrix elements. This theory predicts a universal form of the spectral correlation functions determined solely by some global symmetries of the system (time-reversal invariance and value of the spin). Time-reversal-invariant quantum system are represented by a Gaussian orthogonal ensemble (GOE) of random matrices when the system has rotational symmetry and by a Gaussian symplectic ensemble (GSE) otherwise. Chaotic systems without time reversal invariance are represented by the Gaussian unitary ensemble (GUE). The dimension  $\beta$  of the underlying parameter space is used to label these three ensembles: for GOE, GUE and GSE,  $\beta$  takes the values 1, 2 and 4, respectively. Among several measures representing spectral correlations, the nearest-neighbor level-spacing distribution function  $p(s)$  has been extensively studied so far. According to the random matrix theory, the level spacing distribution function in the chaotic phase is approximated by the Wigner-Dyson distribution, namely,

$$P_\beta(s) = a_\beta s^\beta \exp(-b_\beta s^2). \quad (1)$$

The coefficients  $a_\beta$  and  $b_\beta$  are determined by the normalization conditions  $\int_0^\infty P_\beta(s)ds = \int_0^\infty sP_\beta(s)ds = 1$ , as  $a_1 = \pi/2$ ,  $a_2 = 32/\pi^2$ ,  $a_4 = 2^{18}/36\pi^3$ ,  $b_1 = \pi/4$ ,  $b_2 = 4/\pi$ , and  $b_4 = 64/9\pi$ . For  $s \ll 1$ , the distribution function is proportional to  $s^\beta$ , which implies that adjacent energy levels cannot approach each other indefinitely because of mixing between two extended states.

The assumptions that lead to RMT do not apply for mixed systems. The Hamiltonian of a typical mixed system can be described as a random matrix with some (or all) of its elements as randomly distributed. Here the distributions of various matrix elements need not be same, may or may not be correlated and some of them can be non-random too. This is a difficult route to follow. So far in the literature, there is no rigorous statistical description for the transition from integrability to chaos. There have been several proposals for phenomenological random matrix theories that interpolate between the Wigner-Dyson RMT and banded RM with the (almost) Poissonian level statistics. The standard route of the derivation is to sacrifice basis invariance but keep matrix-element independence. The first work in this direction is due to Rosenzweig and Porter [11]. They model the Hamiltonian of the mixed system by a superposition of two matrices: a diagonal matrix of random elements having the same variance and a matrix drawn from a GOE. Therefore, the variances of the diagonal elements total Hamiltonian are different from those of the off-diagonal ones, unlike the standard GOE Hamiltonian in which the variances of diagonal elements are twice of the off-diagonal ones. Hussein and Sato [12] used the maximum entropy principle to construct such ensembles by imposing additional constraints. Ensembles of band random matrices whose entries are equal to zero outside a band of width  $b$  along the principal diagonal have also been often used to model mixed systems [13, 14, 15, 16, 17].

Another route for generalizing RMT is to conserve base invariance but allow for correlation of matrix elements. This has been achieved by maximizing non-extensive entropies subject to the constraint of fixed expectation value of  $\text{Tr}(H^\dagger H)$  [18, 19, 20, 21, 22, 23, 24]. Recently, an equivalent approach is presented in [25, 26], which is based on the method of superstatistics (statistics of a statistics) proposed by Beck and Cohen [27]. This formalism has been elaborated and applied successfully to a wide variety of physical problems, e.g., in [28, 29, 30, 31, 32, 33, 34, 35]. In thermostatics, superstatistics arises as weighted averages of ordinary statistics (the Boltzmann factor) due to fluctuations of one or more intensive parameter (e.g. the inverse temperature). Its application to RMT assumes the spectrum of a

mixed system is made up of many smaller cells that are temporarily in a chaotic phase. Each cell is large enough to obey the statistical requirements of RMT but has a different distribution parameter  $\eta$  associated with it, according to a probability density  $f(\eta)$ . Consequently, the superstatistical random-matrix ensemble describes the mixed system as a mixture of Gaussian ensembles with a statistical weight  $f(\eta)$ . Therefore one can evaluate any statistic for the superstatistical ensemble by simply integrating the corresponding statistic for the conventional Gaussian ensemble.

## II. NONEXTENSIVE GENERALIZATION OF RMT

In 1957 Jaynes [39] proposed a rule, based on information theory, to provide a constructive criterion for setting up probability distributions on the basis of partial knowledge. This leads to a type of statistical inference which is called the maximum-entropy principle (MaxEnt). It is the least biased estimate possible on the given information. Jaynes showed in particular how his rule, when applied to statistical mechanics, leads to the usual Gibbs' canonical distribution. The core of the MaxEnt method resides in interpreting entropy, through the Shannon axioms, as a measure of the “amount of uncertainty” or of the “amount of information that is missing” in a probability distribution. This was an important step forward because it extended the applicability of the notion of entropy far beyond its original roots in thermodynamics. Balian considered the application of MaxEnt to the random-matrix theory by maximizing the Shannon entropy under constraints of normalization and existence of the expectation value of  $\text{Tr}(\mathbf{H}^\dagger \mathbf{H})$ . In this section, we consider possible generalization of RMT by extremizing two different entropies, namely Tsallis' and Kaniadakis', rather than Shannon's entropy. The extremization is again subject to the constraint of normalization and existence of the expectation value of  $\text{Tr}(\mathbf{H}^\dagger \mathbf{H})$ . For completeness, we start by a brief review of the conventional random-matrix theory.

### A. RMT from Shannon's entropy

Balian [10] derived the weight functions  $P(\mathbf{H})$  for the random-matrix ensembles from MaxEnt postulating the existence of a second moment of the Hamiltonian. He applied the

conventional Shannon definition for the entropy to ensembles of random matrices as

$$S_{\text{Sch}} = - \int d\mathbf{H} P(\mathbf{H}) \ln P(\mathbf{H}) \quad (2)$$

and maximized it under the constraints of normalization of  $P(\mathbf{H})$  and fixed mean value of  $\text{Tr}(\mathbf{H}^\dagger \mathbf{H})$ . The latter constraint ensures basis independence, which is a property of the trace of a matrix. Then, the distribution  $P(H)$  is determined from the extremum of the functional

$$F_{\text{Sch}} = S_{\text{Sch}} - \xi \int d\mathbf{H} P(\mathbf{H}) - \eta \int d\mathbf{H} P(\mathbf{H}) \text{Tr} (H^T H), \quad (3)$$

where  $\xi$  and  $\eta$  are Lagrange multipliers. Its maximum is obtained equating its functional derivative to zero. He obtained

$$P_\beta(\mathbf{H}) = \frac{1}{Z(\eta)} \exp [-\eta \text{Tr} (\mathbf{H}^\dagger \mathbf{H})], \quad (4)$$

where  $Z(\eta) = \int \exp [-\eta \text{Tr} (\mathbf{H}^\dagger \mathbf{H})] d\mathbf{H}$ .

It is easy to see that the joint distribution of matrix elements obtained in Eq. (4) satisfies the two conditions of RMT, namely uncorrelated matrix-elements and Base independence. The first condition follow since the distribution (4) is a Gaussian distribution with inverse variance  $1/2\eta$ , since  $\text{Tr}(\mathbf{H}^\dagger \mathbf{H}) = \sum_{i=1}^N \left(H_{ii}^{(0)}\right)^2 + 2 \sum_{\gamma=0}^{\beta-1} \sum_{i>j} \left(H_{ij}^{(\gamma)}\right)^2$ , where all the four matrices  $H^{(\gamma)}$  with  $\gamma = 0, 1, 2, 3$  are real. This allows the factorization of  $P_\beta(\mathbf{H})$  into products of terms depending only on the individual matrix elements. Therefore, the matrix elements of  $\mathbf{H}$  are independent. Base independence follows from the fact that the distribution (18) depends on  $\mathbf{H}$  in the combination  $\text{Tr}(\mathbf{H}^\dagger \mathbf{H})$ . Indeed, if two matrices  $\mathbf{A}$  and  $\mathbf{B}$  that express the same operator in two different bases are related by a similarity transformation  $\mathbf{B} = \mathbf{T}^{-1} \mathbf{A} \mathbf{T}$ , then such operators have the same trace.

The joint distribution of eigenvalues  $E_i$  immediately follows from Eq. (3). With  $\mathbf{H} = \mathbf{U}^{-1} \mathbf{E} \mathbf{U}$ , where  $\mathbf{U}$  is the global unitary group and  $\mathbf{E} = \text{diag}(E_1, \dots, E_N)$  the volume element  $d\mathbf{H}$  has the form

$$d\mathbf{H} = |\Delta_N(\mathbf{E})|^\beta d\mathbf{E} d\mu(\mathbf{U}), \quad (5)$$

where  $\Delta_N(\mathbf{E}) = \prod_{n>m} (E_n - E_m)$  is the Vandermonde determinant and  $d\mu(\mathbf{U})$  the invariant Haar measure of the unitary group [8, 9]. Integrating over  $\mathbf{U}$  and noting that  $\text{Tr}(\mathbf{H}^\dagger \mathbf{H}) = \text{Tr} \mathbf{E}^2$  yields the joint probability density of eigenvalues in the form

$$P_\beta(\eta, E_1, \dots, E_N) = C_\beta \prod_{n>m} |E_n - E_m|^\beta \exp \left( -\eta \sum_{i=1}^N E_i^2 \right), \quad (6)$$

where  $C_\beta$  is a normalization constant. All of the spectral properties of the Gaussian random-matrix ensemble can be obtained from Eq. (6). However, this is not an easy task. Lacking simple exact results, and guided by the case  $N = 2$ , Wigner proposed a form for the nearest neighbor spacing (NNS) distribution  $p(s)$  of eigenvalues. This “Wigner surmise”, originally stated for  $\beta = 1$ , has the form

$$p_\beta(s, \eta) = \frac{\sqrt{2\mu}}{\Gamma[(\beta+1)/2]} \left(\frac{\eta s}{2}\right)^\beta \exp\left(-\frac{\eta s^2}{2}\right). \quad (7)$$

The parameter  $\eta$  is determined by the condition of unit mean spacing,  $\int_0^\infty s p_\beta(s) ds = 1$ , as

$$\eta = 2 \frac{\Gamma^2[(\beta+2)/2]}{\Gamma^2[(\beta+1)/2]}. \quad (8)$$

Although the Wigner surmise is strictly valid for two-dimensional ensembles, it is an accurate approximation for ensemble with arbitrarily large  $N$ . To demonstrate the accuracy, we expand this distribution for the case of  $\beta = 2$  in powers of  $s$  to obtain

$$p_2(s) = \frac{32}{\pi^2} s^2 \left(1 - \frac{4}{\pi} s^2 + \dots\right) \cong 3.242 s^2 - 4.128 s^4 + \dots, \quad (9)$$

while the power-series expansion of the corresponding exact distribution for ensembles with  $N \rightarrow \infty$  [8] yields

$$p_{2,\text{exact}}(s) = \frac{\pi^2}{3} s^2 - \frac{2\pi^4}{45} s^4 + \dots \cong 3.290 s^2 - 4.329 s^4 + \dots. \quad (10)$$

The Wigner surmise has been successfully applied to the NNS distributions for numerous chaotic systems.

## B. RMT from Tsallis’ entropy

The past decade has witnessed a considerable interest devoted to non-conventional statistical mechanics. Much work in this direction followed the line initiated by Tsallis’ seminal paper [40]. The standard statistical mechanics is based on the Shannon entropy measure  $S = -\sum_i p_i \ln p_i$  (we use Boltzmann’s constant  $k_B = 1$ ), where  $\{p_i\}$  denotes the probabilities of the microscopic configurations. This entropy is extensive. For a composite system  $A + B$ , constituted of two independent subsystems  $A$  and  $B$  such that the probability  $p(A + B) = p(A)p(B)$ , the entropy of the total  $S(A + B) = S(A) + S(B)$ . Tsallis proposed a non-extensive generalization:  $S_q = (1 - \sum_i p_i^q)/(q - 1)$ . The entropic index  $q$  characterizes

the degree of extensivity of the system. The entropy of the composite system  $A + B$ , the Tsallis' measure verifies

$$S_q(A + B) = S_q(A) + S_q(B) + (1 - q)S_q(A)S_q(B), \quad (11)$$

from which the denunciation non-extensive comes. Therefore,  $S_q(A + B) < S_q(A) + S_q(B)$  if  $q > 1$ . This case is called sub-extensive. If  $q < 1$ , the system is in the super-extensive regime. The standard statistical mechanics recovered for  $q = 1$ . Applications of the Tsallis formalism covered a wide class of phenomena; for a review please see, e.g. [41].

The Tsallis entropy is defined for the joint matrix-element probability density  $P_\beta(q, \mathbf{H})$  by

$$S_q [P_{\text{Ts},\beta}(q, \mathbf{H})] = \left( 1 - \int dH [P_{\text{Ts},\beta}(q, \mathbf{H})]^q \right) / (q - 1). \quad (12)$$

We shall refer to the corresponding ensembles as the Tsallis orthogonal ensemble (TsOE), the Tsallis Unitary ensemble (TsUE), and the Tsallis symplectic ensemble (TsSE). For  $q \rightarrow 1$ ,  $S_q$  tends to Shannon's entropy, which yields the canonical Gaussian orthogonal, unitary or symplectic ensembles (GOE, GUE, GSE) [8, 10].

There are more than one formulation of non-extensive statistics which mainly differ in the definition of the averaging. Some of them are discussed in [42]. We apply the most recent formulation [43]. The probability distribution  $P_{\text{Ts},\beta}(q, \mathbf{H})$  is obtained by maximizing the entropy under two conditions, where  $\sigma_\beta$  is a constant. The optimization of  $S_q$  with these constraints yields a power-law type for

$$P_\beta(q, H)P_{\text{Ts},\beta}(q, \mathbf{H}) = \tilde{Z}_q^{-1} [1 + (q - 1)\tilde{\eta}_q \{ \text{Tr}(\mathbf{H}^\dagger \mathbf{H}) - \sigma_\beta^2 \}]^{-\frac{1}{q-1}}, \quad (13)$$

where  $\tilde{\eta}_q > 0$  is related to the Lagrange multiplier  $\eta$  associated with the constraint of fixed  $\text{Tr}(\mathbf{H}^\dagger \mathbf{H})$  by

$$\tilde{\eta}_q = \eta / \int d\mathbf{H} [P_{\text{Ts},\beta}(q, \mathbf{H})]^q, \quad (14)$$

and

$$\tilde{Z}_q = \int d\mathbf{H} [1 + (q - 1)\tilde{\eta}_q \{ \text{Tr}(\mathbf{H}^\dagger \mathbf{H}) - \sigma_\beta^2 \}]^{-\frac{1}{q-1}}. \quad (15)$$

It turns out that the distribution (13) can be written hiding the presence of  $\sigma_\beta^2$  in a more convenient form

$$P_{\text{Ts},\beta}(q, \mathbf{H}) = Z_q^{-1} [1 + (q - 1)\eta_q \text{Tr}(\mathbf{H}^\dagger \mathbf{H})]^{-\frac{1}{q-1}}, \quad (16)$$



where

$$\eta_q = \frac{\eta}{\int d\mathbf{H} [P_\beta(q, \mathbf{H})]^q + (1-q)\eta\sigma_\beta^2}, \quad (17)$$

and

$$Z_q = \int d\mathbf{H} [1 + (q-1)\eta_q \text{Tr}(\mathbf{H}^\dagger \mathbf{H})]^{-\frac{1}{q-1}}. \quad (18)$$

The probability density  $P_{\text{Ts},\beta}(q, \mathbf{H})$  depends on  $\mathbf{H}$  through  $\text{Tr}(\mathbf{H}^\dagger \mathbf{H})$  and is therefore invariant under arbitrary rotations in the matrix space. This ensures base invariance. It decays by a power law as the square of any matrix element tends to infinity in contrast with the Gaussian decay of the distribution function of the conventional random matrix ensembles.

We now calculate the joint probability density for the eigenvalues of the Hamiltonian  $\mathbf{H}$ . Expressing the volume element in the matrix-element space in the form (5) and integrating over the "angular" variables, one obtains

$$P_{\text{Ts},\beta}(\eta_q, E_1, \dots, E_N) = C_{\text{Ts},\beta} \prod_{n>m} |E_n - E_m|^\beta \left[ 1 + (q-1)\eta_q \sum_{i=1}^N E_i^2 \right]^{-\frac{1}{q-1}}, \quad (19)$$

where  $C_{\text{Ts},\beta}$  is a normalization constant.

In order to obtain a generalization of Wigner's surmise, we consider the special case of  $N = 2$ . In this case,

$$P_{\text{Ts},\beta}(\eta_q; \varepsilon, s) = c_{\text{Ts},\beta} s^\beta \left[ 1 + (q-1)\eta_q \left( 2\varepsilon^2 + \frac{1}{2}s^2 \right) \right]^{-\frac{1}{q-1}}, \quad (20)$$

where  $\varepsilon = (E_1 + E_2)/2$ ,  $s = |E_1 - E_2|$ . For this case, the distribution (20) has to be complemented by the auxiliary condition that the quantity inside the square bracket has to be positive. We here consider the case of  $q \geq 1$  where no limitations are imposed on the values of the variables  $\varepsilon$  and  $s$ , and refer the reader interested in the other case of  $q < 1$  to Ref. [21]. The NNS distribution is obtained by integrating (20) over  $\varepsilon$  from  $-\infty$  to  $\infty$ .

$$p_{\text{Ts},\beta}(q, s) = a_{\text{Ts},\beta} s^\beta [1 + b_{\text{Ts},\beta} s^2]^{-\frac{1}{q-1} + \frac{1}{2}}, \quad (21)$$

where  $a_{\text{Ts},\beta}$  is a normalization coefficient and  $b_{\text{Ts},\beta}$  is obtained for the requirement of unit mean spacing. Explicitly,

$$a_{\text{Ts},\beta} = \frac{2b_{\text{Ts},\beta}^{(\beta+1)/2} \Gamma\left(\frac{1}{q-1} - \frac{1}{2}\right)}{\Gamma\left(\frac{\beta+1}{2}\right) \Gamma\left(\frac{1}{q-1} - \frac{\beta}{2} - 1\right)} \quad \text{and} \quad b_{\text{Ts},\beta} = \frac{\Gamma^2\left(\frac{\beta+2}{2}\right) \Gamma^2\left(\frac{1}{q-1} - \frac{\beta}{2} - \frac{3}{2}\right)}{\Gamma^2\left(\frac{\beta+1}{2}\right) \Gamma^2\left(\frac{1}{q-1} - \frac{\beta}{2} - 1\right)} \quad (22)$$

The second moment of the distribution  $\langle s^2 \rangle = \int_0^\infty s^2 p_\beta(s) ds$  is given by

$$\langle s^2 \rangle = \frac{\Gamma\left(\frac{\beta+1}{2}\right) \Gamma\left(\frac{\beta+3}{2}\right) \Gamma\left(\frac{1}{q-1} - \frac{\beta}{2} - 1\right) \Gamma\left(\frac{1}{q-1} - \frac{\beta}{2} - 2\right)}{\Gamma^2\left(\frac{\beta+2}{2}\right) \Gamma^2\left(\frac{1}{q-1} - \frac{\beta}{2} - \frac{3}{2}\right)} \quad (23)$$

It diverges unless  $q < q_\infty = 1 + 2/(\beta + 4)$ , which is equal to 1.40, 1.33 and 1.25 for the orthogonal, unitary and symplectic ensemble, respectively. This imposes physical bound on the admissible values of  $q$ , because  $\langle s^2 \rangle$  has to be finite in order to force condition that  $\text{Tr}(\mathbf{H}^\dagger \mathbf{H})$  has to be finite. At higher values of the entropic index, non-extensive statistics does not apply to the random matrix model. The peak of the distribution in Eq. (21) is located at  $s_\beta = \sqrt{\beta/b_{\text{Ts},\beta} [-1 - \beta + 1/(1 - q)]}$ . It moves from  $s_1 = 0.798$  to  $s_1 = 0.368$ , from  $s_2 = 0.886$  to  $s_2 = 0.408$  and from  $s_4 = 0.940$  to  $s_4 = 0.671$  as  $q = 1$  (the standard Wigner's surmise) to  $q_\infty$ . Neither reaches 0, the peak position of the Poisson distribution  $\exp(-s)$  of the integrable systems. The proposed non-extensive ensemble in the three cases of  $\beta = 1, 2$  and 4 evolve the shape predicted by the corresponding Wigner surmise towards the Poisson distribution, but never reach it.

### C. RMT from Kaniadakis' entropy

In this section, we consider a possible generalization of RMT based on an extremization of Kaniadakis'  $\kappa$ -entropy [44, 45, 46]. This entropy shares the same symmetry group of the relativistic momentum transformation and has applications in cosmic-ray and plasma physics. For the matrix-element probability distribution function, it reads

$$S_\kappa [\kappa, P_{\text{K},\beta}(\kappa, \mathbf{H})] = -\frac{1}{2\kappa} \int dH \left( \frac{\alpha^\kappa}{1 + \kappa} [P_{\text{K},\beta}(\kappa, \mathbf{H})]^{1+\kappa} - \frac{\alpha^{-\kappa}}{1 - \kappa} [P_{\text{K},\beta}(\kappa, \mathbf{H})]^{1-\kappa} \right) \quad (24)$$

with  $\kappa$  a parameter with value between 0 and 1; the case of  $\kappa = 0$  corresponds to the Schannon entropy. Here,  $\alpha$  is a real positive parameter. Kaniadakis has considered two choices of  $\alpha$ , namely  $\alpha = 1$  and  $\alpha = Z$ , where  $Z$  is the generalized partition function. We here adopt the second choice. The matrix-element distribution  $P_{\text{K},\beta}(\kappa, \mathbf{H})$  is obtained by extremizing the functional

$$F_K = S_\kappa - \eta_K \int dH P_{\text{K},\beta}(\kappa, \mathbf{H}) \text{Tr}(\mathbf{H}^\dagger \mathbf{H}), \quad (25)$$

where  $\eta_K$  is a Lagrange multiplier. One arrives to the following distribution

$$P_{\text{K},\beta}(\kappa, \mathbf{H}) = \frac{1}{Z_\kappa} \exp_{\{\kappa\}} [-\eta_K \text{Tr}(\mathbf{H}^\dagger \mathbf{H})], \quad (26)$$

where

$$Z_\kappa = \int d\mathbf{H} \exp_{\{\kappa\}} \left[ -\eta_K \text{Tr} (\mathbf{H}^\dagger \mathbf{H}) \right]. \quad (27)$$

Here  $\exp_{\{\kappa\}}[x]$  is the  $\kappa$ -deformed exponential [44] which is defined by

$$\exp_{\{\kappa\}}[x] = \left( \sqrt{1 + \kappa^2 x^2} + \kappa x \right)^{1/\kappa} = \exp \left( \frac{1}{\kappa} \text{arcsinh } \kappa x \right). \quad (28)$$

Concerning the asymptotic behavior of  $P_{K,\beta}(\kappa, \mathbf{H})$  we easily obtain that

$$P_{K,\beta}(\kappa, \mathbf{H}) \sim \left| \kappa \eta_K \text{Tr} (\mathbf{H}^\dagger \mathbf{H}) \right|^{-1/|\kappa|} \quad (29)$$

as the square of any of the matrix elements tends to infinity

The probability density  $P_K(\kappa, \mathbf{H})$  depends on  $\mathbf{H}$  through  $\text{Tr}(\mathbf{H}^\dagger \mathbf{H})$  and is therefore invariant under arbitrary rotations in the matrix space. Using Eq. (26) and integrating over the "angular variables"  $\mathbf{U}$  yields the joint probability density of eigenvalues in the form

$$P_{K,\beta}(\kappa; E_1, \dots, E_N) = C_{K,\beta} \prod_{n>m} |E_n - E_m|^\beta \exp_{\{\kappa\}} \left[ -\eta_K \sum_{i=1}^N E_i^2 \right]. \quad (30)$$

where  $C_{K,\beta}$  is a normalization constant.

In order to obtain a generalization of the Wigner surmise, we consider the case of two-dimensional random-matrix ensemble where  $N = 2$  and  $n = 3$  and restrict our consideration to the orthogonal ensemble with  $\beta = 1$ . In this case, Eq. (30) reads

$$P_{K,\beta}(\kappa; \varepsilon, s) = \frac{2(1 + 3\kappa/4)}{B\left(\frac{1}{2\kappa} - \frac{3}{4}, \frac{3}{2}\right)} (\kappa \eta_K)^{3/2} s \exp_{\{\kappa\}} \left[ -\eta_K \left( 2\varepsilon^2 + \frac{1}{2}s^2 \right) \right], \quad (31)$$

where  $\varepsilon = (E_1 + E_2)/2$ ,  $s = |E_1 - E_2|$ , and  $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$  is the Beta function [47]. The NNS distribution is obtained by integrating (31) over  $\varepsilon$  from  $-\infty$  to  $\infty$ . This can be done by changing the variable  $\varepsilon$  into  $x = \exp[-\frac{1}{\kappa} \text{arcsinh}(\kappa \eta_K s^2/2)]$ , integrating by parts, and then replacing the variable  $x$  by another new variable,  $y = \exp(\kappa x)$ . The resulting integral can be solved by using the following identity [47]

$$\int_u^\infty y^{-\lambda} (y + \beta)^\nu (y - u)^{\mu-1} dy = u^{\mu+\nu-\lambda} B(\lambda - \mu - \nu, \mu) {}_2F_1 \left( -\nu, \lambda - \mu - \nu; \lambda - \nu; -\frac{\beta}{u} \right), \quad (32)$$

for  $|\beta/u| < 1$  and  $0 < \mu < \lambda - \nu$ , where  ${}_2F_1(\nu, \mu; \lambda; x)$  is the hypergeometric function. Thus, after straightforward calculations we can express the NNS distribution as

$$p_{K,1}(\kappa, s) = -2 \left(1 + \frac{3}{4}\kappa\right) \eta_K s e^{(1/2-1/\kappa)\text{arcsinh}(\kappa\eta_K s^2/2)} \frac{B\left(\frac{1}{\kappa} - \frac{1}{2}, \frac{3}{2}\right)}{B\left(\frac{1}{2\kappa} - \frac{3}{4}, \frac{3}{2}\right)} {}_2F_1\left(-\frac{1}{2}, \frac{1}{\kappa} - \frac{1}{2}; \frac{1}{\kappa} + 1; -e^{-2\text{arcsinh}(\kappa\eta_K s^2/2)}\right). \quad (33)$$

The condition of unit mean spacing defines the quantity  $\eta_K$  as

$$\eta_K = \left[ \frac{\pi k^{3/2} \left(1 + \frac{3}{4}\kappa\right)}{(1 - \kappa^2) B\left(\frac{1}{2\kappa} - \frac{3}{4}, \frac{3}{2}\right)} \right]^2. \quad (34)$$

We note that the function  $B\left(\frac{1}{\kappa} - \frac{1}{2}, \frac{3}{2}\right)$  diverges at  $\kappa = \kappa_c = 1/2$ , which serves as an upper limit for the range of variation of  $\kappa$ . We also note that the mean square spacing diverges at  $\kappa = \kappa_\infty = 2/5$ .

### III. SUPERSTATISTICAL GENERALIZATION OF RMT

Let us first recall the basic idea underlying superstatistics. We will then proceed to construct a generalization of RMT in the spirit of superstatistics.

#### A. Beck and Cohen's superstatistics

Consider a complex system in a nonequilibrium stationary state. Such a system will be, in general, inhomogeneous in both space and time. Effectively, it may be thought to consist of many spatial cells, in each of which there may be a different value of some relevant intensive parameter, e.g. the inverse temperature  $\beta$ . Beck and Cohen [27] assumed that this quantity fluctuates adiabatically slowly, namely that the time scale is much larger than the relaxation time for reaching local equilibrium. In that case, the distribution function of the non-equilibrium system consists in Boltzmann factors  $\exp(-\beta H)$  that are averaged over the various fluctuating inverse temperatures

$$F(H) = \int g(\beta) \frac{\exp(-\beta H)}{z(\beta)} d\beta \quad (35)$$

where  $z(\beta)$  is a normalizing constant, and  $g(\beta)$  is the probability distribution of  $\beta$ . Let us stress that  $\beta^{-1}$  is a local variance parameter of a suitable observable, the Hamiltonian of

the complex system in this case. Ordinary statistical mechanics are recovered in the limit  $g(\beta) \rightarrow \delta(\beta)$ . In contrast, different choices for the statistics of may lead to a large variety of probability distributions  $F(H)$ . Several forms for  $g(\beta)$  have been studied in the literature, e.g. [27, 32, 48]. In particular, Tsallis statistics is generated from Eq. (35) if  $\beta$  is a chi-squared random variable

$$g(\beta) = \frac{1}{\Gamma(\nu/2)} \left( \frac{\nu}{2\beta_0} \right)^{\nu/2} \beta^{\nu/2-1} e^{-\nu\beta/2\beta_0} \quad (36)$$

where  $\beta_0$  is the mean value. A chi-squared distribution derives from the summation of squares of  $\nu$  Gaussian random variables  $X_1, \dots, X_\nu$  due to various relevant degrees of freedom in the system, where the  $X_i$  are independent, and  $\langle X_i \rangle = 0$ . If  $\beta^{-1}$  rather than  $\beta$  is the sum of several squared Gaussian random variables, the resulting distribution  $g(\eta)$  is the inverse  $\chi^2$  distribution given by

$$g(\beta) = \frac{\beta_0}{\Gamma(\nu/2)} \left( \frac{\nu\beta_0}{2} \right)^{\nu/2 - \nu/2 - 2} \beta e^{-\nu\beta_0/2\beta}, \quad (37)$$

where again  $\beta_0$  is the average of  $\beta$ . Instead of being a sum of many contributions, the random variable  $\beta$  may be generated by multiplicative random processes. Then  $\ln \beta = \sum_{i=1}^{\nu} \ln X_i$  is a sum of Gaussian random variables. Thus it is log-normally distributed,

$$g(\beta) = \frac{1}{\sqrt{2\pi v\beta}} e^{-[\ln(\beta/\mu)]^2/2v^2}, \quad (38)$$

which has an average  $\mu\sqrt{w}$  and variance  $\mu^2 w(w-1)$ , where  $w = \exp(v^2)$ .

## B. RMT within superstatistics

To apply the concept of superstatistics to RMT, one assumes the spectrum of a (mixed) system as made up of many smaller cells that are temporarily in a chaotic phase. Each cell is large enough to obey the statistical requirements of RMT but is associated with a different distribution of the parameter  $\eta$  in Eq. (4) according to a probability density  $f(\eta)$ . Consequently, the superstatistical random-matrix ensemble used for the description of a mixed system consists of a superposition of Gaussian ensembles. Its joint probability density distribution of the matrix elements is obtained by integrating the distribution given in Eq. (4) over all positive values of  $\eta$  with a statistical weight  $f(\eta)$ ,

$$P(H) = \int_0^\infty f(\eta) \frac{\exp[-\eta \text{Tr}(H^\dagger H)]}{Z(\eta)} d\eta. \quad (39)$$

Despite the fact that it is hard to make this picture rigorous, there is indeed a representation which comes close to this idea [49, 50].

Beck, Cohen and Swinney [48] proposed the derivation of superstatistics starting from time-series. The idea is that superstatistical thermostatics results as a convolution of two statistics, one characterized by the Boltzmann factor and the other corresponding to inverse-temperature fluctuations. This requires the existence of two relaxation times. A justification for the use of the above-mentioned superstatistical generalization of RMT in the study of mixed systems, is given in [51]. It is based on the representation of their energy spectra in the form of discrete time series in which the level order plays the role of time. The representation of the suitably transformed eigenvalues of a quantum system as a time series has recently allowed to determine the degree of chaoticity of the dynamics of the system [52, 53, 54, 55, 56]. Reference [51] considers two billiards with mushroom-shaped boundaries as representatives of systems with mixed regular-chaotic dynamics and three with the shape of Limaçon billiards, one of them of chaotic and two of mixed dynamics. The quantum eigenvalues and statistical properties of the eigenfunctions were obtained experimentally by exploiting the equivalence of the Schrödinger equation of a plane quantum billiard and the Helmholtz equation for the electric field strength in a cylindrical microwave resonator for wave lengths longer than twice the height of the resonator. The billiards with mixed dynamics have classical phase spaces of different structures for the two families of billiards. The "time-series" analysis of their spectra indeed manifests the existence of two relaxation lengths. The short one, which is defined as the average length over which energy fluctuations are correlated, is of the order of the mean level spacing. The long one, which is by an order of magnitude larger, characterizes the typical linear size of the heterogeneous domains of the total spectrum.

The new framework of RMT provided by superstatistics should now be clear. The local mean spacing is no longer uniformly set to unity but allowed to take different (random) values at different parts of the spectrum. The parameter  $\eta$  is no longer a fixed parameter but it is a stochastic variable with probability distribution  $f(\eta)$ . The observed mean level spacing is just the expectation value of the local ones. The fluctuation of the local mean spacing is due to the correlation of the matrix elements, which disappears for chaotic systems. In the absence of these fluctuations,  $f(\eta) = \delta(\eta - \eta_0)$  and we obtain the standard RMT. Within the superstatistics framework, we can express any statistic  $\sigma(E)$  of a mixed system that can

in principle be obtained from the joint eigenvalue distribution by integration over some of the eigenvalues, in terms of the corresponding statistic  $\sigma^{(G)}(E, \eta)$  for a Gaussian random ensemble. The superstatistical generalization is given by

$$\sigma(E) = \int_0^\infty f(\eta) \sigma^{(G)}(E, \eta) d\eta. \quad (40)$$

The remaining task of superstatistics is the computation of the distribution  $f(\eta)$ , which has been introduced in Eq. (39). The time series analysis in Ref. [51] allows to derive the same parameter distribution  $f(\eta)$ . The obtained distribution agrees better with the inverse  $\chi^2$  distribution given by Eq. (37) rather than the other two distributions (36) and (38). We have already mentioned that the  $\chi^2$  distribution of the superstatistical parameter  $\eta$  yields Tsallis statistics for RMT, which is considered in the previous section. The log-normal distribution does not lead to simple analytical results for the important level statistics like the NNS distribution. For these reasons, we shall confine our further consideration to the case of inverse  $\chi^2$  distributed superstatistical parameter  $\eta$ .

### C. Superstatistical generalization of Wigner's surmise

The superstatistics generalization of NNS distribution for an ensemble belonging to a given symmetry class is obtained by substituting the NNS distribution of the corresponding Gaussian ensemble  $p_\beta(s, \eta)$  for  $\sigma^{(G)}(E, \eta)$  in (40) and integrating over the local mean level spacing  $\eta$

$$p_{\text{SS},\beta}(s) = \int_0^\infty f(\eta) p_\beta(s, \eta) d\eta. \quad (41)$$

For an inverse  $\chi^2$  distribution of  $\eta$ , given by Eq. (37), one obtains the following superstatistical NNS distribution

$$p_{\text{SS},\beta}(\nu, s) = \frac{4\sqrt{\eta_0/\nu}}{\Gamma(\frac{\nu}{2}) \Gamma(\frac{1+\beta}{2})} (\sqrt{\eta_0\nu}s/2)^{\frac{1+\nu+\beta}{2}} K_{\frac{1+\nu-\beta}{2}}(\sqrt{\eta_0\nu}s), \quad (42)$$

where  $K_m(x)$  is a modified Bessel function [47] and  $\eta_0$  again is determined by the requirement that the mean-level spacing  $\langle s \rangle$  equals unity,

$$\eta_0 = \frac{16\pi}{\nu^3} \left[ \frac{\Gamma(\frac{3+\nu}{2}) \Gamma(1 + \frac{\beta}{2})}{\Gamma(\frac{\nu}{2}) \Gamma(\frac{1+\beta}{2})} \right]^2. \quad (43)$$

The inverse  $\chi^2$  distribution of  $\eta$  follows when the quantity  $\eta^{-1}$  is the sum of  $\nu$  squared Gaussian random variables. If we take this assumption literally, we must restrict  $\nu$  to take

positive integer values. As the transition from integrability to chaos is known to proceed continuously, we have to relax this condition and allow  $\nu$  to take any real value between 1 and  $\infty$ . Let us restrict our following consideration to the case of orthogonal symmetry with  $\beta = 1$ . Using the asymptotic expression of the modified Bessel function [47], we easily find

$$\lim_{\nu \rightarrow \infty} p_{\text{SS},1}(\nu, s) = \frac{\pi}{2} s e^{-\pi s^2/4}, \quad (44)$$

which is the Wigner surmise, as required. The other limit of  $\nu \rightarrow 1$  yields the semi-Poisson distribution

$$p_{\text{SemiPoisson}}(s) = 4s e^{-2s}, \quad (45)$$

which is known to provide a satisfactory description for the spectra of pseudointegrable systems such as planar polygonal billiards, when all their angles are rational with  $\pi$  [57]. The motion of the corresponding classical systems in phase space is not restricted to a torus like for integrable systems, but to a surface with a more complicated topology [58]. We therefore conclude that the assumption that the inverse square of the variance of matrix elements as an inverse  $\chi^2$  variable allows one to model the transition out of chaos (corresponding to  $\nu \gg 1$ ) until the system reaches the state of quasi-integrability as the effective number  $\nu$  of components of  $\eta^{-1}$  approaches 1. If one allows  $\nu$  to take lower values, then the distribution (42) tends to the Poisson distribution as  $\nu \rightarrow -1$ ;

$$p_{\text{SS},1}(-1, s) = e^{-s}. \quad (46)$$

We there conclude that formula (42) can provide a model for describing the stochastic transition all the way from integrability to chaos passing by the stage of quasi-integrability.

#### IV. COMPARISON WITH NUMERICAL EXPERIMENT

The NNS distributions  $p_{\text{Ts},\beta}(q, s)$  and  $p_{\text{K},\beta}(\kappa, s)$  obtained above when the entropy is given by the Tsallis and Kaniadakis entropies, respectively, as well as the superstatistical distribution  $p_{\text{SS},\beta}(\nu, s)$  describe evolution of the spacing distribution from the Wigner shape to the Poissonian. They can be useful for describing systems with mixed regular-chaotic at least when more familiar distributions such as Berry and Robnik's or Brody's distribution [61, 62] do not fit the data satisfactorily. We shall demonstrate this by using these distributions to fit the NNS distribution of levels of a number of mixed systems.



### A. Mushroom billiards

Billiards can be used as simple models in the study of Hamiltonian systems. They consist of a point particle which is confined to a container of some shape and reflected elastically on impact with the boundary. The shape determines whether the dynamics inside the billiard is regular, chaotic or mixed. The best-known examples of chaotic billiards are the Sinai billiard (a square table with a circular barrier at its center) and the Bunimovich stadium (a rectangle with two circular caps) [59]. Neighboring parallel orbits diverge when they collide with dispersing components of the billiard boundary. In mixed billiards, some neighboring parallel orbits converge at first, but divergence prevails over convergence on average. Divergence and convergence are balanced in integrable billiards such as circles and ellipses.

Recently Bunimovich introduced the so-called ‘mushroom’ billiard [60] with the novel feature of a well-understood divided phase-space comprising a single integrable region and a single ergodic one. We restrict ourselves here to mushroom billiards which consist of a semicircular region, the ‘hat’ and a ‘stem’, which is symmetrically attached to its base. As the width of the stem varies from zero to the diameter of the hat, there is a continuous transition from integrability (the semicircle billiard) to ergodicity (in case of a rectangular stem the stadium billiard). In mushroom billiards, the regular region has a well-defined semicircular border. It is composed of those trajectories in the hat that never cross this border and therefore remain in the hat forever. Their integrability is due to the conservation of the reflection angle for collisions with the semicircular boundary. The chaotic component consists of trajectories that enter the stem of the mushroom billiard. Two mushroom billiards have been recently investigated experimentally [63]. The ratio of the width of the stem to the diameter of the hat is 1:3 for the small mushroom billiard and 2:3 for the large. Both billiards have mixed dynamics with classical phase spaces of different structures for the two billiards. The degree of chaos, which is the measure of all chaotic parts of the phase space, is 45.5 % and 82.9 %, respectively. Both systems have been studied experimentally in the quantum limit exploiting the analogy between a quantum billiard and a flat cylindric microwave billiard of the same shape. The electromagnetic resonances in each microwave cavity can directly be associated with quantum states in the corresponding quantum billiard. For the evaluation of statistical measures, a sufficiently large number of resonances is needed.

The first 780 resonances could be detected in the small billiards and 938 in the large one. The quantum eigenvalues were obtained experimentally by exploiting the equivalence of the Schrödinger equation of a plane quantum billiard and the Helmholtz equation for the electric field strength in a cylindrical microwave resonator for wave lengths longer than twice the height of the resonator. To compare the statistical properties of the eigenvalues with universal predictions considered in the present paper, they have to be rescaled to unit mean spacing. This is done by an unfolding procedure using Weyl's formula [64], which relates the billiard area and circumference to the number of resonance frequencies below a given one.

We compared the resulting NNS distributions given in Eqs. (21), (33) and (42) with the experimental ones for the mushroom billiards and the two Limaçon billiards with mixed dynamics. In Fig. 1 the experimental results for  $p_1(s)$  are shown by histogram together while the distributions obtained by starting with the Tsallis and Kaniadakis entropies and the superstatistical distributions are shown by the dashed, dashed-dotted and solid lines, respectively. The best fit values of the parameters are given in Table I, together with the  $\chi^2$  values calculated by

$$\chi^2 = \frac{1}{N} \sum_{i=1}^N [p_1(s_i) - p_{X,1}(s_i)]^2, \quad (47)$$

where  $N$  is the number of experimental spacings and  $X$  stands for Ts (Tsallis), K (Kaniadakis) or SS (superstatistics).

## B. Random binary networks

As another example of mixed systems, we consider a numerical experiment by Gu et al. [65] on a random binary network. Impurity bonds are employed to replace the bonds in an otherwise homogeneous network. In such a network, there exist a lot of geometric resonances randomly distributing in the resonant area. Based on the Green's-function formalism, the eigenvalues of Green's-matrix are solved, the sequence of which forms the resonance spectrum. The authors of Ref. [65] numerically calculated more than 700 resonances for each sample. For each impurity concentration  $p$ , they considered 1000 samples with totally more than 700 000 levels computed. Their results for four values of concentration  $p$  are compared with both the Tsallis, Kaniadakis and superstatistical NNS distributions in Fig. 2. The best fit values of the parameters as well as the corresponding  $\chi^2$  value are given in Table II. The high

statistical significance of the data allows us to assume the advantage of the superstatistical distributions for describing the results of this experiment, as compared to the other two distribution families.

## V. CONCLUSION

Random matrix theory is the statistical theory of random matrices, whose entries fluctuate as independent Gaussian random numbers. The matrix-element distribution is obtained by extremizing Shannon's entropy subject to the constraint of normalization and constant trace of the square of the matrix. The latter constraint renders the matrix-element distribution base independent. While most of the previously proposed generalization of RMT violate base invariance, the ones reviewed in this paper conserve it. Non-extensive generalizations extremize nonextensive entropies such as Tsallis' or Kaniadakis', rather than Shannon's. Superstatistical generalizations, on the other hand, allow the fluctuation of the mean local density of states which is fixed in the standard theory. These generalizations of RMT, seen from the present perspective, may bear interest per se because of the additional nontrivial fluctuations introduced in a simple model. In addition, they may constitute a useful statistical paradigm for the analysis of the spectral fluctuations of systems with mixed regular-chaotic dynamics. For this purpose, simple analytical expressions are derived in each case for the nearest neighbor level distributions, being among the most popular characteristics of spectral fluctuation. The formalism has been checked by the analysis of experimental resonance spectra of mixed microwave billiards and geometrical resonances in random binary networks. The predictions of the three models satisfactorily describe the experimental trends of the evolution of NNS distributions during the transition out of chaos. The considered experimental data agree better in most cases with the corresponding distributions predicted by the superstatistical approach when the fluctuating parameter has an inverse  $\chi^2$  distribution.

- 
- [1] A.J. Lichtenberg, M.A. Lieberman, Regular and Stochastic Motion, Applied Mathematical Sciences (Springer, New York, 1983).
  - [2] C. Chandre, H.R. Jauslin, Phys. Rep. 365, 1 (2002).

- [3] J.-P. Eckmann and D. Ruelle, *Rev. Mod. Phys.* **67**, 617 (1985).
- [4] S. S. E. H. Elnashaie and S. S. Elshishini, *Dynamical Modelling, Bifurcation and Chaotic Behavior of Gas-Solid Catalytic Reactions* (Gordon & Breach, Amsterdam, 1996).
- [5] L. A. Bunimovich and S. Venkatuyiri, *Phys. Rep.* **290**, 81 (1997).
- [6] M.V. Berry, *J. Phys. A* **10**, 2083 (1977).
- [7] O. Bohigas, M.J. Giannoni, and C. Schmit, *Phys. Rev. Lett.* **52**, 1 (1984).
- [8] M.L. Mehta, *Random Matrices* 2nd ed. (Academic, New York, 1991).
- [9] T. Guhr, A. Müller-Groeling, and H. A. Weidenmüller, *Phys. Rep.* **299**, 189 (1998).
- [10] R. Balian, *Nuovo Cim.* **57**, 183 (1958).
- [11] N. Rosenzweig and C. E. Porter, *Phys. Rev.* **120**, 1698 (1960).
- [12] M. S. Hussein and M. P. Sato, *Phys. Rev. Lett.* **70**, 1089 (1993); *Phys. Rev. C* **47**, 2401 (1993).
- [13] A. Casati, L. Molinari, and F. Izrailev, *Phys. Rev. Lett.* **64**, 1851 (1990).
- [14] Y. V. Fyodorov and A. D. Mirlin, *Phys. Rev. Lett.* **67**, 2405 (1991).
- [15] A.D. Mirlin, Y.V. Fyodorov, F.M. Dittes, J. Quezada, and T.H. Seligman, *Phys. Rev. E* **54**, 3221 (1996).
- [16] V.E. Kravtsov, K.A. Muttalib, *Phys. Rev. Lett.* **79**, 1913 (1997).
- [17] F. Evers and A.D. Mirlin, *Phys. Rev. Lett.* **84**, 3690 (2000); *Phys. Rev. B* **62**, 7920 (2000).
- [18] J. Evans and F. Michael, e-prints cond-mat/0207472 and /0208151.
- [19] F. Toscano, R. O. Vallejos, and C. Tsallis, *Phys. Rev. E* **69**, 066131 (2004).
- [20] F. D. Nobre and A.M. C. Souza, *Physica A* **339**, 354 (2004).
- [21] A. Y. Abul-Magd, *Phys. Lett. A* **333**, 16 (2004).
- [22] A. C. Bertuola, O Bohigas, and M. P. Prato, *Phys. Rev. E* **70**, 065102(R) (2004).
- [23] A.Y. Abul-Magd, *Phys. Rev. E* **71**, 066207 (2005).
- [24] A.Y. Abul-Magd, *Phys. Lett. A* **361**, 450 (2007).
- [25] A.Y. Abul-Magd, *Physica A* **361**, 41 (2006).
- [26] A.Y. Abul-Magd, *Phys. Rev. E* **72**, 066114 (2005).
- [27] C. Beck and E. G. D. Cohen, *Physica A* **322**, 267 (2003).
- [28] E. G. D. Cohen, *Physica D* **193**, 35 (2004).
- [29] C. Beck, *Physica D* **193**, 195 (2004).
- [30] C. Beck, *Europhys. Lett.* **64**, 151 (2003).
- [31] F. Sattin and L. Salasnich, *Phys. Rev. E* **65**, 035106(R) (2003).

- [32] F. Sattin, Phys. Rev. E 68, 032102 (2003).
- [33] A. Reynolds, Phys. Rev. Lett. 91, 084503 (2003).
- [34] M. Ausloos and K. Ivanova, Phys. Rev. E 68, 046122 (2003).
- [35] C. Beck, Physica A 331, 173 (2004).
- [36] S. Abe, Phys. Rev. E 66, 046134 (2002).
- [37] A. M. C. Souza and C. Tsallis, Phys. Lett. A 319, 273 (2003).
- [38] C. Tsallis and A. M. C. Souza, Phys. Rev. E 67, 026106 (2003); Phys. Lett. A 319, 273 (2003).
- [39] E.T. Jaynes, Phys. Rev. 106 (1957) 620; 108 (1957) 171.
- [40] C. Tsallis, J. Stat. Phys. 52, 479 (1988).
- [41] C. Tsallis, Lect. Notes Phys. 560, 3 (2001).
- [42] Q. A. Wang, Eur. Phys. J. B 26, 357 (2002).
- [43] C. Tsallis, R. S. Mendes, and A. R. Plastino, Physica A 261, 534 (1998).
- [44] G. Kaniadakis, Physica A 296 (2001) 405.
- [45] G. Kaniadakis, Phys. Rev. E 66 (2002) 056125.
- [46] G. Kaniadakis, Phys. Rev. E 72 (2005) 036108.
- [47] Gradshteyn and I.M. Ryzhik, Table of integrals, series and products, Academic Press, New York, 1980.
- [48] C. Beck, E. G. D. Cohen, and H. L. Swinney, Phys. Rev. E 72, 026304 (2005).
- [49] G. Le Caër and R. Delannay, Phys. Rev. E **59**, 6281 (1999).
- [50] K. A. Muttalib and J. R. Klauder, Phys. Rev. E **71**, 055101(R) (2005).
- [51] A.Y. Abul-Magd, B. Dietz, T. Friedrich, and A. Richter, Phys. Rev. E 77, 046202 (2008).
- [52] A. Relaño, J. M. G. Gómez, R. A. Molina, J. Retamosa, and E. Faleiro, Phys. Rev. Lett. **89**, 244102 (2002); E. Faleiro, J. M. G. Gómez, R. A. Molina, L. Muñoz, A. Relaño, and J. Retamosa, Phys. Rev. Lett. **93**, 244101 (2004); E. Faleiro, U. Kuhl, R. A. Molina, L. Muñoz, A. Relaño, and J. Retamosa, Phys. Lett. A **358**, 251 (2006); R.A. Molina, J. Retamosa, L. Muñoz, A. Relaño, and E. Faleiro, Phys. Lett. B **644**, 25 (2007).
- [53] J. M. G. Gómez, A. Relaño, J. Retamosa, E. Faleiro, L. Salasnich, M. Vraničar, and M. Robnik, Phys. Rev. Lett. **94**, 084101 (2005).
- [54] M. S. Santhanam and J.N. Bandyopadhyay, Phys. Rev. Lett. **95**, 114101 (2005).
- [55] P. Manimaran, P. A. Lakshmi, and P. K. Panigrahi, Phys. Rev. E **72** 046120 (2005); J. Phys. A **39**, L599 (2006).

- [56] M. S. Santhanam, J. N. Bandyopadhyay, and D. Angom, Phys. Rev. E **73**, 015201 (2006).
- [57] E. Bogomolny, U. Gerald, and C. Schmit, Phys. Rev. E 59, R1315 (1999).
- [58] P. J. Richens and M.V. Berry, Physica (Amsterdam) 2D, 495 (1981).
- [59] L. Bunimovich, Funct. Anal. Appl. **8**, 254 (1974).
- [60] L.A. Bunimovich, Chaos **11** (2001).
  
- [61] M. V. Berry and M. Robnik, J. Phys. A **17**, 2413 (1984).
- [62] T. Brody, Lett. Nuovo Cim. **7**, 482 (1973).
- [63] B. Dietz, T. Friedrich, M. Miski-Oglu, A. Richter, and F. Schäfer, Phys. Rev. E **75**, 035203(R) (2007).
- [64] H. P. Baltes and E. R. Hilf, *Spectra of Finite Systems* (Bibliographisches Institut Mannheim, 1975).
- [65] Y. Gu, K. W. Yu, and Z. R. Yang, Phys. Rev. E 65, 046129 (2002).

TABLE I. Best-fit parameters for the experimental NNS distribution of resonances in the small and large mushroom billiards. The corresponding  $\chi^2$  values are given in parentheses.

Distribution	Small billiard	Large billiard
Tsallis	$q = 1.336$ (0.0189)	$q = 1.221$ (0.0031)
Kaniadakis	$\kappa = 0.423$ (0.0159)	$\kappa = 0.017$ (0.0069)
Superstatistical	$\nu = -0.441$ (0.0026)	$\nu = 2.31$ (0.0018)

TABLE II. Best-fit parameters for the numericcal-experimental NNS distribution of geometrical resonances in the binary random network with different impurity concentrations  $p$ . The corresponding  $\chi^2$  values are given in parentheses.

Distribution	$p = 0.1$	$p = 0.2$	$p = 0.3$	$p = 0.4$
Tsallis	$q = 1.380$ (0.0034)	$q = 1.322$ (0.0022)	$q = 1.263$ (0.0015)	$q = 1.219$ (0.0009)
Kaniadakis	$\kappa = 0.444$ (0.0232)	$\kappa = 0.421$ (0.0067)	$\kappa = 0.398$ (0.0059)	$\kappa = 0.012$ (0.0051)
Superstatistical	$\nu = -0.188$ (0.0021)	$\nu = 0.617$ (0.0002)	$\nu = 1.76$ (0.0002)	$\nu = 3.12$ (0.0003)

FIG.1 NNS distributions of resonances in mushroom billiards (histograms), measure by Friedrich et al. [63] compared with the Tsallis (solid lines), Kaniadakis (dashed) and super-statistical (dashed-dotted) NNS distributions. The best-fit parameters are given in Table I.



FIG.2 NNS distributions of geometrical resonances in random network (dots), calculated by Gu et al. [65] compared with the Tsallis (solid lines), Kaniadakis (dashed) and super-statistical (dashed-dotted) NNS distributions. The best-fit parameters are given in Table II.



